What was Greek about Greek Mathematics?

Richard Wallace

An odd anecdote frequently told about the philosopher Aristippus, the well-known Socratic and hedonist, is very revealing about Greek attitudes to mathematics:

> Once when Aristippus was undertaking a voyage, his ship was wrecked and he was cast ashore at Syracuse. The first thing which caused him to regain his courage was the sight of a geometrical diagram in the sand. For he reasoned that he had come among Greeks and wise men, not barbarians.

This particular version of the story is from Galen,[1] but there are many others. Latin writers who tell the story, like Cicero[2] and Vitruvius[3] rather spoil the point of the story by making Aristippus cry out that he saw traces of *men* (*hominum vestigia video*), presumably because neither Cicero nor Vitruvius was Greek. The change spoils the anecdote because there are many sights which could have told him that men lived nearby, but the point of the original story was surely that geometry was something essentially and intrinsically Greek; or, to put it in another way, civilised.

The reason that the story is odd (besides the improbability of attributing an admiration of mathematics to a notorious hedonist who, according to Aristotle, compared the subject unfavourably with other trades because, unlike them, mathematics "had nothing to say about good or bad"[4]) is what it does not explain. Why is mathematics Greek? What was different about Greek mathematics? And why did they do it?

It is generally agreed that mathematics is among the greatest, and is perhaps *the* greatest, of the achievements of the Greeks; certainly it is the one with the most direct influence today. What is more, it is one of their most characteristic achievements, marked by those qualities on which the Greeks so prided themselves, and which they so rarely displayed — order, limit, harmony, and rationality. If, however, by mathematics we mean the development of techniques for manipulating numbers and figures, then the Greeks were clearly by no means the first or only people to develop mathematical systems; there was a long history of the use of sophisticated techniques among their neighbours (not to mention the Chinese), and virtually all of the mathematics necessary for *practical* purposes was already available when Greek mathematics comes on the scene. The principal contribution of the Greeks to the development of mathematics is usually (and rightly) seen to be the concept of formal proof, and as we

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1 Galen *Protr.* 5.

2 Cicero *De Rep.* 1.29.

3 Vitruvius *De Arch.* 6.1.

4 Aristotle *Metaph.* 2.996a35.

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learn more and more about the mathematical capabilities of their predecessors, the importance of this contribution becomes even clearer. I would argue that there is one further contribution which the Greeks made, namely the establishment of mathematics as a cultural phenomenon, something the educated person must know, an art rather than a technique. The two are linked. It can be argued that the geometrical theorem is best thought of as a literary form, analogous to poetry and other types of literature, rather than a practical method of expounding useful techniques. It is worth noting that not all Greek mathematics is in this form; Diophantus' algebra, for example, is, like much non-Greek mathematics, in the form of worked examples with no sign of any formal proof at all. The sort of proof proffered by the traditional style of geometrical work — a statement of axioms or unproved (and presumably unprovable) assertions[5] followed by a sequence of theorems, one building on another until the goal is reached — is largely an illusion once we get beyond Euclid (whose work is not essentially creative but a systematisation of what had gone before, and for that reason very systematic). If the more complex works of (for example) Archimedes really did what they seem to do, namely prove everything, they would be very long indeed (and in any case proving everything would be quite an impossible project). Archimedes refers to propositions of Euclid only a couple of times,[6] and these references are usually regarded as glosses for the very good reason that if he had wanted to quote a reference every time he used a proposition proved elsewhere, he could have done so much more often (without actually adding anything to the clarity or comprehensibility of his proofs). There is an element of showmanship in the best Greek mathematics, especially that of Archimedes who, unlike Euclid, does not (at any rate on a superficial reading) go through a logical and systematic progression of theorems but announces his results at the beginning of a treatise (in the preface) and then sets off on a series of harmless-looking propositions the purpose of which is not always immediately clear; on these he builds even more complex propositions until, sometimes to the reader's surprise, he produces the results for which he was looking rather like rabbits out of a conjurer's hat. In other words, it is a performance, and it is what makes Archimedes one of the most exciting of all mathematicians to read. It must be said, however, that it is possible to have a perfectly good system of mathematics without this kind of formal proof (unless we choose to define mathematics in a way which incorporates the concept of formal proof), and most cultures which have taken an interest in mathematics do without it (or at any rate the predecessors of the Greeks and a number of cultures uninfluenced by them managed, without using Greek-style proofs, to explore quite complex relationships which it would be perverse not to call mathematical). What is also obvious is that the Greeks themselves did not *discover* geometric propositions by these means. Archimedes frequently distinguishes the discoverer of a proposition from the

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5 They are variously classified. Euclid's *Elements* begins with *horoi* (= definitions), *aitēmata* (= postulates), and *koinai ennoiai* (= axioms); Archimedes *On the Sphere and Cylinder* begins with *axiōmata* (= definitions) and *lambanomena* (= postulates).

6 E.g. Archimedes *Sph. Cyl.* 1.6.

one who proved it; the formal theorem is (among other things) an expository device, a literary form.

Greek mathematics seems to have taken what we now regard as its characteristically distinctive form in the 4th century BCE. What happened then was not just an extension in mathematical knowledge but the creation of mathematics as we know it, including the concept of proof and the establishment of a new status for mathematics as a cultural, rather than a practical, enterprise.

The nature of mathematics before this change took place is best illustrated by some examples. In the late 5th century Hippocrates of Chios wrote *On the Quadrature of Lunes* in which he demonstrates that a number of areas bounded by arcs of circles can be shown to be equal to various rectilinear figures. It is mentioned by several writers on mathematics, but an extended account is given by Simplicius,[7] drawing on Eudemus' history of mathematics. To work, the quadratures need the proposition proved by Euclid 12.2, that circles are to one another as the square on their diameters, from which it is possible to show that similar segments of circles are to one another as the squares on their bases, which is what Hippocrates needs. But in Euclid this proposition is proved by the method conventionally known as the method of exhaustion, and it is hard to see how it could be proved any other way. This method, however, seems to have been devised first by Eudoxus; certainly Archimedes[8] attributes to Eudoxus two theorems proved by exhaustion in Euclid 12, and no-one attributes the invention of the method to Hippocrates or any of his predecessors. There is no evidence that anyone before Eudoxus used it, and yet it is a very powerful method with many applications, so it is inconceivable that it should have been available before Eudoxus but left no trace in mathematics. By normal standards, then, (or at any rate by the standards subsequently exemplified in the works of Euclid), what Hippocrates is offering is not a proof. Simplicius knows this; he refers to Eudemus' *hypomnēmatikon tropon*, "note form" and tells us that this was the *archaikon ethos*, the way they did it in the old days. In other words, Simplicius was used to formal proofs of the kind with which we are familiar from Euclid and Archimedes, sees gaps in the arguments he finds in Eudemus and assumes that it is simply a summary, but he then tells us that this is what proofs were like in the old days (and it is indeed difficult to see how there could be a complete proof of Hippocrates' propositions without the use of the method of exhaustion). A second example of a similar kind is Proclus'[9] claim that Thales *proved* (*apodeixai,* the usual word for a mathematical proof) that a circle is bisected by its diameter. This can indeed be proved, and Pappus does so, but Euclid does not, he takes it as given as part of Definition 17 (the passage on which Proclus is commenting). It seems reasonable to assume that Euclid is trying to reduce to a minimum the number of unproved assertions on which his work is based, and that as a consequence he does not regard this as something which can securely be proved (though he does need it), so it seems inescapable that whatever Thales was doing with this proposition it did not amount to anything which we

7 Simplicius in *Phys. A2* 60.22-68.32.

8 Archimedes *Sph. Cyl.* Preface; *Eratosth.* Preface.

9 Proclus in *Euc.* 1.157.Def.17.

(following Euclid) would recognise as a mathematical proof. Archimedes'[10] report that certain properties of cones and pyramids were *discovered* by Democritus but *proved* by Eudoxus seems to confirm the view that formal strict mathematical proof was a relatively late development. What is more, the notorious difficulties in interpreting the mathematical passage at the beginning of Plato's *Theaetetus*[11] arise in part from Plato's use of non-standard vocabulary, which suggests that the passage reflects an early stage in the systematisation of the subject.

If mathematics as we understand it really was a development of the 4th century BCE, we have to ask what was the context in which this change took place, and what the mathematicians who brought it about thought they were doing. Three notorious passages are relevant here. The first is from Xenophon's *Memorabilia*:[12]

> He [Socrates] said that geometry should be learnt up to the point where you could, if necessary, measure accurately in receiving or handing out or sharing land, or in marking out some piece of work. That was so easy to learn that anyone who applied his mind to the measurement could easily know how big the ground was and go away knowing how it was measured. But he disapproved of taking geometry as far as diagrams which are hard to understand. He said that he did not see what the use of it was, though he was not unfamiliar with it. He said that these things could waste away a man's life, and prevent him from learning many useful subjects.

The other two are from Isocrates, the first from the *Antidosis*:[13]

> Most people think of such subjects [astronomy, geometry, and eristic arguments] as idle talk and pettifogging, for none of them has any use for personal or public business. ... I do not agree with this, but my opinion is not far from it. ... Those who spend their time chattering about the niceties of astronomy and geometry and are forced to apply their minds to subjects which are difficult to learn also become accustomed to speaking and to taking pains over their words and their arguments and to keeping their minds from wandering and so are, through the exercise and stimulus they receive in these disciplines, able to absorb and learn more easily and more quickly the subjects which are better and more valuable. I do not think it right to give the name of philosophy to something which is of no immediate use to us either in speech or in action; I call it a sort of mental gymnastics, a preparation for philosophy. It is more adult than what children learn at school, but is much the same sort of thing. For when children have worked through grammar and music and the rest of the school syllabus they have still made no progress in speaking or making decisions in practical matters, but are in a better position to learn these greater and more important subjects. So I would advise the young to spend some time on these subjects, but not to allow their minds to be wasted away by them.

The second passage of Isocrates is from the *Panathenaicus*:[14]

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10 Archimedes *Eratosth.* Preface.

11 Plato *Tht.* 147d.

12 Xenophon *Mem.* 4.7.

13 Isocrates *Antid.* 261-268.

14 Isocrates *Pan.* 26-28.

> Not only do I not look down on the education handed down by our ancestors, I even commend that which has been established in our own days; I mean geometry and astronomy and the so-called eristic dialogues [*dialogous tous eristikous kaloumenous*] which the young take more pleasure in than they should, though all of the older generation would say that they are intolerable. Nevertheless I urge those with an inclination to this sort of thing to work hard and apply themselves to it, on the grounds that even if it does them no good it keeps the young away from many other vices. ... But I do not think that these studies are suitable for adults and those who have come of age. For I see that some of those who have learnt these things so thoroughly that they actually teach them do not use effectively the knowledge which they have, and in the general business of life have less sense than their students, I hesitate to say, than their slaves.

It seems reasonable to assume that all three passages are contributions to a contemporary argument (and we can surely discount Xenophon's claim that Socrates knew all about modern mathematics as being on a par with the prescience of Plato's Socrates in philosophical problems). If so, then it is clear that the comments are a response to something new, an innovation, what Isocrates calls "*tēn eph'hēmōn paideian*". It is interesting to note just what strikes them about the new mathematics. For Xenophon it is about *dussunetōn grammatōn*, incomprehensible diagrams, and for him this is what distinguishes it from the good old mathematics which you use to measure fields; this may tell us more about Xenophon's mathematical abilities than about contemporary mathematics, but it does suggest a subject which has just gone into another gear. For Isocrates, what strikes him is its *akribeia*, its attention to detail, perhaps, or we might interpret it in terms of the precision of its results. This was of course the aspect of mathematics which Plato found significant, though Isocrates is more impressed by its capacity to help cultivate a proper pettifogging attitude. In either case it is the close argument used by mathematicians which differentiates it from other disciplines.

Two other points are made. First, there is a great attack on the alleged uselessness of the subject. This is the same point which Aristotle reports was made by Aristippus[15] when he rejected it because it said nothing *peri agathōn kai kakōn,* referring presumably to concrete goods and ills rather than ethical principles; at any rate this is what is suggested by the fact that the other arts to which he compares mathematics unfavourably are things like building and leather-working. Interestingly, even Plato, who was a strong advocate of the value of mathematics as an educational tool for directing the mind away from the world towards higher realities, does from time to time show a marked unease about its lack of practical applications, and slips in remarks about its usefulness in marshalling armies[16] and so on. What emerges from all of this is a contrast between the old mathematics (useful, practical, simple, and boring) and the new geometry which was intellectually demanding, rigorous, and exciting, but one cannot quite

15 Aristotle *ibid.* (Cf. n. 4 *supra*).
16 E.g. Plato *R.* 7, 522a-e.

see what the purpose of it all is beyond a vague feeling that it might do you good.

The second point is its attraction to the young. To a modern reader there is something wildly implausible about the idea of mathematics as a candidate for the rôle of an attractive pursuit to keep young men off the streets and out of trouble. It is however a recurring theme among writers of the period. Both Aristotle[17] and Plato[18] regard mathematics as something specifically suitable for the young. The dubiously Platonic *Erastai*[19] opens with a scene of the "best of the young men" (*tōn neōn tous epieikestatous dokountas einai*) completely engrossed in an argument (*mala espoudakote*) about Anaxagoras and Oinopides. Despite the fact that Proclus[20] classifies these two as mathematicians (entirely on the basis of this passage), there seems little doubt that the discussion is about astronomy (the participants are drawing circles and indicating angles with their hands), but it is astronomy of a very mathematical kind, in that the movements of the heavenly bodies are being treated in geometrical terms. In any case, the fact that the new mathematics was attacked is itself evidence that it was popular. Why would anybody bother to attack a subject in which the young took no interest? The phenomenon requires explanation. I suggest that the clue is to be found in the way Isocrates puts mathematics, astronomy, and *eristikoi logoi* in the same indiscriminate category. Ancient mathematics tended to be problem-centred. That in itself is not surprising; so is modern mathematics. In ancient mathematics, however, problems are particularly prominent. For example, Archimedes' treatises announce themselves not as explorations of particular relationships but as attempts to prove certain challenging propositions.[21] Archimedes is particularly significant, since he is virtually the only creative mathematician whose work survives to any extent (although it is possible that Euclid's *Elements* may contain original work, in its format and approach the work is, so far as one can tell, presented as a systematisation or compilation; the case of Apollonius of Perge's *Conic Sections* is not so clear, though Apollonius himself claims originality for only part of it). Like all aspects of Greek life, mathematics seems to have been extremely competitive. There are various stories about problems like the so-called "Delian problem" (the duplication of the cube) being approached competitively. Eutocius[22] gives us an implausible report of Plato gathering his students together and setting it to them. None of this need be taken seriously, but it is interesting that what the argument seems to be about is not just whether the proofs work and how elegant they are, but what counts as a proof. Plutarch[23] represents Plato as berating his colleagues for using instruments and mechanical solutions (*organikas kai mēchanikas kataskeuas*) on the grounds that resorting to sensible things rather than eternal truths is to ruin

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17 E.g. Aristotle *EN* 6.8.1142a11-20.

18 E.g. Plato *R.* 536d.

19 Plato *Amat.* 132a-b.

20 Proclus *in Euc.1* 65.21-66.4 Friedlein.

21 E.g. Archimedes *Sph Cyl.* 1, Preface.

22 Eutocius *Comm in Archim. de Sphaera et Cyl.* 2. 88.4-90.13 Heiberg.

23 Plutarch *Quaest. Conviv.* 8.

the virtues of geometry (*diaphtheiresthai to geōmetrias agathon*). Curiously Eutocius[24] attributes to Plato just such a mechanical solution as Plutarch decries. The issue seems to be whether the method is sufficiently rigorous. This argument seems never really to have been settled. Archimedes' treatise *The Method* uses techniques (about which some modern mathematicians believe he is unduly apologetic) based on concepts like centre of gravity, to solve a series of problems, but he does not seem to regard them as complete proofs himself (it is not entirely clear why; Archimedes himself simply says that "investigation by this method does not provide proofs";[25] precisely why he thought it does not is a matter for speculation) and gives more conventional proofs of the same propositions in other treatises.

In considering the history of mathematics we naturally tend to concentrate on the expansion of the subject, the devising of proofs for more and more complex relationships. But what seems to be happening here is a much more fundamental discussion on what mathematics is and what counts as a mathematical proof. In other words, we are seeing the creation of mathematics as we know it. It is also revealing that we find Archimedes circulating propositions which he has proved *without* their proofs, and challenging other mathematicians to show that they are as good as he is by providing the missing proofs.[26] Indeed, in the preface to *On Spirals* he says that among the propositions which he had circulated before releasing this particular set of proofs he had slipped in a couple which he believed could not be proved, to catch out the cheats who claimed that they had found proofs when they had not. Propositions and proofs seem not to have always gone together (and I have already mentioned another case where Archimedes[27] refers to a proposition which was discovered by one person but proved by another). It looks as if what we are looking at is a culture of competitive problem-solving, which perhaps gives us a context in which mathematics could become of absorbing interest to the young. It may be that the closest modern analogy to the situation of 4th-century mathematics is the enthusiasm of young people for the computer, which leads a proportion of them to become totally absorbed in a discipline which is arid, in many cases useless, perhaps even dehumanising, and which would fill them with horror if it were to be a compulsory part of the school syllabus, but which brings status, is "modern" (i.e. their parents do not understand it) and which is closely enough related to the uncontroversially educational to make the adult world at least unsure of its attitude to it.

By Aristotle's time Greek mathematicians were well on the way to developing a geometry which was not only in content but also in form close to that with which we are familiar from Euclid. For example, Aristotle not only knows a good number of Euclid's axioms, he knows them in a form similar to that in which Euclid gives them. So Euclid's Axiom 3, *ean apo isōn isa aphairethē, ta kataleipomena estin isa*, appears in Aristotle's *Metaphysics* 10 1061b4 as *apo*

24 Eutocius *Comm. in Archim. de Sphaera et Cyl.* 56.13-58.14 Heiberg.

25 Archimedes *Eratosth.* Preface.

26 See, for example, the prefaces to *Eratosth.* and to *Sph. Cyl.* 1 & 2.

27 Archimedes *Eratosth.* Preface.

tōn isōn isōn aphairethentōn isa ta leipomena. By then the transition from the old unsystematic mathematics seems to have been completed. The driving force behind the change seems to have been competitive problem solving, the competition being not only in the discovery of solutions, but also in the style of solutions. But who set the agenda?

For some time now historians of mathematics have tended to treat mathematics as if it were largely autonomous. This was a reaction against an earlier view (going back to antiquity) which saw the subject as parasitic on philosophy (seeing, for example, the method of exhaustion as an attempt to solve the difficulties raised by Zeno's paradoxes). It has also been normal to see the cultural significance of mathematics as lying in its providing a paradigm of what proof is which in turn helped to form philosophers' concept of knowledge. Perhaps this is not quite the whole truth. The concept of mathematical proof cannot be found securely before the 4th century, and if this is so we must surely be looking at a much more complex relationship between the two disciplines of philosophy and mathematics. At any rate, it looks as if mathematics is not a semi-detached part of Greek culture, but springs from the intensively competitive nature of Greek society, the Greek's tendency to turn everything into a race, which made mathematics into a contest where you win not just by the ingenuity of your own proofs, but by probing and undermining the proofs of others. So the home of real mathematics, as of real philosophy, will perhaps turn out to be the *eristikos dialogos*.

Keele University